Math 210B Lecture 27 Notes

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1 Ideals of Extensions of Rings

1.1 The going up theorem

Suppose B/A is an extension of commutative rings. How do ideals of A and ideals of B compare? If we have an ideal $\mathfrak{a}$ of A, then $\mathfrak{a}B$ is an ideal of B. We can go back by sending $\mathfrak{b} \mapsto \mathfrak{f} \cap A$.

Definition 1.1. We say an ideal $\mathfrak{b} \subseteq B$ **lies over** $\mathfrak{a} \subseteq A$ if $\mathfrak{b} \cap A = \mathfrak{a}$.

If $\mathfrak{p}$ is prime, then $\mathfrak{p}B$ need not be prime.

Example 1.1. Extend $\mathbb{Z}$ to $\mathbb{Z}[\sqrt{2}]$. Then $(2) \mapsto 2\mathbb{Z}[\sqrt{2}] = (sqrt2)^2$. However, if $\mathfrak{q} \subseteq \mathbb{Z}[\sqrt{2}]$ is prime, then $\mathfrak{q} \cap \mathbb{Z}$ is prime in $\mathbb{Z}$.

Proposition 1.1. *Let B/A be an extension of commutative rings.*

1. *If $\mathfrak{b} \subseteq B$ lies over $\mathfrak{a} \subseteq A$, then $A/\mathfrak{a}$ injects into $B/\mathfrak{b}$.*
2. *If $S \subseteq A$ is a multiplicatively closed subset and B/A is integral, then so is $S^{-1}B/S^{-1}A$.*
3. *If B/A is integral and A is a field, then so is B.*

Proposition 1.2. *Suppose B/A is integral. If $\mathfrak{p} \subseteq A$ is prime, then there exists a prime $\mathfrak{q} \subseteq B$ lying over $\mathfrak{p}$.*

Proof. Consider $S_\mathfrak{p} = A \setminus \mathfrak{p}$. Let $B_\mathfrak{p} := S_\mathfrak{p}^{-1}B$; this is integral over $A_\mathfrak{p}$. Let $\mathfrak{M} \subseteq B_\mathfrak{p}$ be maximal. Then $\mathfrak{m} = \mathfrak{M} \cap A_\mathfrak{p}$ is maximal: $A/\mathfrak{m} \to B/\mathfrak{M}$ is an injection, so by the 1st property, $A/\mathfrak{m}$ is a field. So $\mathfrak{p} = A_\mathfrak{p}$. Let $\iota : B \to B_\mathfrak{p}$. Then $q = \iota^{-1}(\mathfrak{M})$, so $\mathfrak{q}$ is prime. Then $\mathfrak{q} \cap A = \iota^{-1}(\mathfrak{M}) \cap A = \iota^{-1}(A_\mathfrak{p})\iota^{-1}(\mathfrak{p}A_\mathfrak{p}) = \mathfrak{p}$. $\square$

Theorem 1.1 (going up theorem)**.** *Let B/A be integral. Let $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ be primes of A, and let $\mathfrak{q}_1 \subseteq B$ be lying over $\mathfrak{p}_1$. Then there exists a prime $\mathfrak{q}_2 \subseteq B$ with $\mathfrak{q}_2 \supseteq \mathfrak{q}_1$ such that $\mathfrak{q}_2$ lies over $\mathfrak{p}_2$.*

Proof. Let $\overline{A} = A/\mathfrak{p}_1$, and let $\overline{B} = B/\mathfrak{q}_1$. Let $\pi : B \to \overline{B}$ be the quotient map. Let $\overline{\mathfrak{p}_2} := \pi(\mathfrak{p}_2)$. $\overline{B}/\overline{A}$ is integral, so there exists aprime $\overline{\mathfrak{q}_2}$ of $\overline{B}$ lying over $\overline{\mathfrak{p}_2}$. Then $q_2 = \pi^{-1}(\overline{\mathfrak{q}_2}) \supseteq \mathfrak{q}_1$. Then $\mathfrak{q}_2 \cap A = \pi^{-1}(\overline{q}_2 \cap \overline{A}) = \pi^{-1}(\overline{\mathfrak{p}_2}) = \mathfrak{p}_2$ since $\mathfrak{p}_2 \supseteq \mathfrak{p}_1$. $\square$

1.2 The going down theorem

Proposition 1.3. *Let B/A be an extension, and let B' be the integral closure of A in B. Then for any multiplicatively closed $S \subseteq A$, $S^{-1}B'$ is the integral closure of $S^{-1}A$ in $S^{-1}B$.*

That is, integral closure is preserved by localization.

Proof. If $b/s \in S^{-1}B$ is integral over $S^{-1}A$, there exists a monic $f \in S^{-1}A[x]$ $f(b/s) = 0$. Write $f = x^n + \sum_{i=0}^{n-1} \frac{a_i}{s_i} x^i$ with $a_i \in A$, $s_i \in S$. Set $t = s_0 \cdots s_{n-1}$. Then $(st)^n f(x/ts) \in A[x]$ has root $x = bt \in B'$. So $s^{-1}b = s^{-1}t^{-1}x$ in $S^{-1}B'$. □

In commutative algebra, we often study what properties are local. For example, we showed earlier that a module is zero iff its localizations at all maximal or all prime ideals are zero.

Proposition 1.4. *Let A be an integral domain. The following are equivalent.*

1. *A is integrally closed.*
2. *$A_{\mathfrak{p}}$ is integrally closed for all prime ideals $\mathfrak{p} \subseteq A$.*
3. *$A_{\mathfrak{m}}$ is integrally closed for all maximal ideals $\mathfrak{m}$ of A.*

Proof. Let $\overline{A}$ be the integral closure of A in $Q(A)$. Then $A = \overline{A}$ iff $\overline{A}/A = 0$. This is an A-modules, so this happens iff $(\overline{A}/A)_{\mathfrak{p}} = 0$ for all $\mathfrak{p}$. Observe that $(\overline{A}/A)_{\mathfrak{p}} = \overline{A}_{\mathfrak{p}}/A_{\mathfrak{p}}$, where $\overline{A}_p = S_{\mathfrak{p}}^{-1}A$ is the integral closure of $A_{\mathfrak{p}}$. □

Theorem 1.2 (going down theorem)**.** *Let B/A be an integral extension of integral domains such that A is integrally closed. Let $\mathfrak{p}_2 \subseteq \mathfrak{p}_1$ be primes of A, and let $\mathfrak{q}_1 \subseteq B$ be lying over $\mathfrak{p}_1$. Then there exists a prime $\mathfrak{q}_2 \subseteq B$ with $\mathfrak{q}_2 \subseteq \mathfrak{q}_1$ such that $\mathfrak{q}_2$ lies over $\mathfrak{p}_2$.*

1.3 Integral extensions in extensions of the quotient field

Let A be an integral domain, and let $K = Q(A)$. Let L be a finite, separable extension of K, and let B be the integral closure of A in L. Then

Lemma 1.1.

$$\mathrm{Tr}_{L/K}(B) \subseteq A, \qquad N_{L/K}(B) \subseteq A.$$

Proof. The minimal polynomial f of $\beta \in B$ lies in $A[x]$. Then $f = x^n - \mathrm{Tr}_{L/K}(\beta)x^{n-1} + \cdots + (-1)^{n-1}N_{L/K}(\beta)$. □

Proposition 1.5. *There exists an ordered basis $\{\alpha_1, \ldots, \alpha_n\}$ of L/K contained in B^n. Set $d = D(\alpha_1, \ldots, \alpha_n)$ and $M = \sum_{i=1}^n A\alpha_i$. Then $M \subseteq B \subseteq d^{-1}M$.*

Proof. Start with a basis $\{\beta_1, \dots, \beta_n\}$ of L/K. Recall that each $\beta_i = b_i/a_i$ with $b_i \in B$ and $a_i \in A$. So multiplying through by $a_1, \dots, a_n$, we have a basis of L/K in B^n.

Given $\{\alpha_1, \dots, \alpha_n\}$, any $\beta \in L$ has the form $\beta = \sum_{i=1}^n c_i \alpha_i$, where $c_i \in K$. Suppose $\mathrm{Tr}_{L/K}(\alpha\beta)]inA$ for all $\alpha \in B$ (e.g. this holds if $\beta \in B$ by the lemma). Consider $A \ni \mathrm{Tr}_{L/K}(\alpha_i\beta) = \sum_{j=1}^n c_j \mathrm{Tr}_{L/K}(\alpha_i\alpha_j)$. Note that $\mathrm{Tr}_{L/K}(\alpha_i\alpha_j)$ is the $(i,)$ entry of $Q = (\mathrm{Tr}_{L/K}(\alpha_i\alpha_j))$. Then $Q^* = \mathrm{adj}(Q)$, and $QQ^* = dI_n$. So we get

$$QQ^* \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} dc_1 \\ \vdots \\ dc_n \end{bmatrix} \in A^n.$$

So we get $d\beta = d\sum_{i=1}^n a_i\alpha_i = \in \sum_{i=1}^n A\alpha_i = M$. Then $dB \subseteq M$, so $B \subseteq d^{-1}M$. $\square$

Remark 1.1. If B is Noetherian, then M is a finitely generated torsion-free B-submodule of L. If B were a PID, then we would get that M is free.

Now assume K/Q is a finite extension. We could define $\mathrm{disc}(K) = \mathrm{disc}(\text{basis of } O_K/\mathbb{Z})$. This is actually independent of basis.